Problem 20

(a) If $f_0(x) = \dfrac{1}{2-x}$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$, find an expression for $f_n(x)$ and use mathematical induction to prove it.

(b) Graph f_0, f_1, f_2, f_3 on the same screen and describe the effects of repeated composition.

Solution

Write out the first few formulas using the given definitions and try to find a pattern.

$$f_0(x) = \frac{1}{2-x}$$

$$f_1(x) = f_{0+1} = f_0 \circ f_0 = f_0(f_0(x)) = \frac{1}{2 - \left(\frac{1}{2-x}\right)} = \frac{1}{\frac{2(2-x)-1}{2-x}} = \frac{2-x}{3-2x}$$

$$f_2(x) = f_{1+1} = f_0 \circ f_1 = f_0(f_1(x)) = \frac{1}{2 - \left(\frac{2-x}{3-2x}\right)} = \frac{1}{\frac{2(3-2x)-(2-x)}{3-2x}} = \frac{3-2x}{4-3x}$$

$$f_3(x) = f_{2+1} = f_0 \circ f_2 = f_0(f_2(x)) = \frac{1}{2 - \left(\frac{3-2x}{4-3x}\right)} = \frac{1}{\frac{2(4-3x)-(3-2x)}{4-3x}} = \frac{4-3x}{5-4x}$$

$$f_4(x) = f_{3+1} = f_0 \circ f_3 = f_0(f_3(x)) = \frac{1}{2 - \left(\frac{4-3x}{5-4x}\right)} = \frac{1}{\frac{2(5-4x)-(4-3x)}{5-4x}} = \frac{5-4x}{6-5x}$$

$$\vdots$$

$$f_n(x) = \frac{(n+1)-nx}{(n+2)-(n+1)x}$$

In order to prove this, use the principle of mathematical induction. Start by showing that the base case is true: If $n = 0$, then

$$f_0(x) = \frac{(1)-(0)x}{(2)-(1)x} = \frac{1}{2-x}.$$

Now assume the inductive hypothesis, that is,

$$f_k(x) = \frac{(k+1)-kx}{(k+2)-(k+1)x},$$

for some positive integer k. The aim is to show that

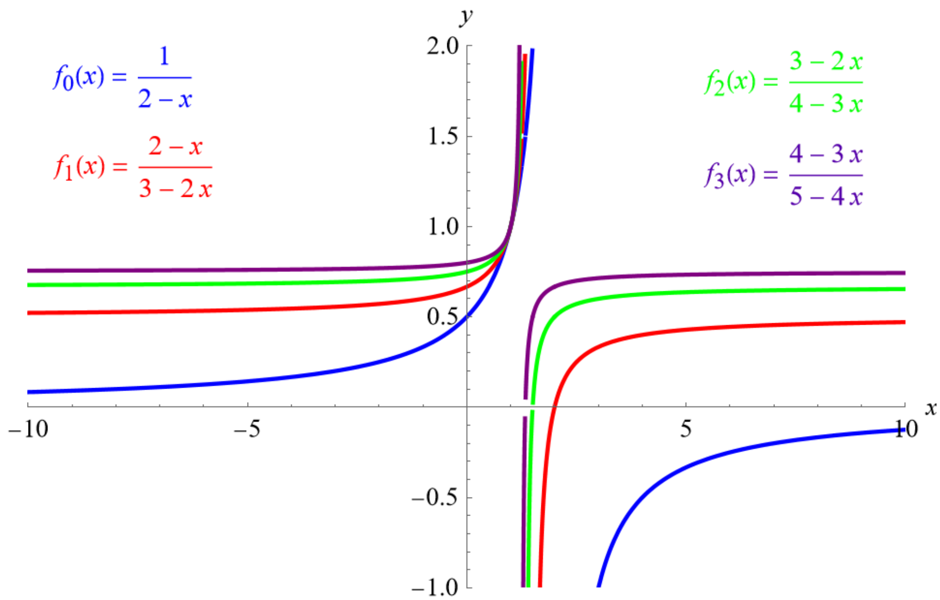
$$f_{k+1}(x) = \frac{[(k+1)+1]-(k+1)x}{[(k+1)+2]-[(k+1)+1]x}.$$

We have

$$\begin{aligned}
f_{k+1}(x) &= f_0 \circ f_k \\
&= f_0(f_k(x)) \\
&= \frac{1}{2 - \frac{(k+1)-kx}{(k+2)-(k+1)x}} \\
&= \frac{1}{\frac{2[(k+2)-(k+1)x]-[(k+1)-kx]}{(k+2)-(k+1)x}} \\
&= \frac{(k+2)-(k+1)x}{2[(k+2)-(k+1)x]-[(k+1)-kx]} \\
&= \frac{(k+2)-(k+1)x}{2(k+2)-(k+1)-[2(k+1)-k]x} \\
&= \frac{(k+2)-(k+1)x}{(k+3)-(k+2)x} \\
&= \frac{[(k+1)+1]-(k+1)x}{[(k+1)+2]-[(k+1)+1]x}.
\end{aligned}$$

Therefore, by the principle of mathematical induction,

$$f_n(x) = \frac{(n+1)-nx}{(n+2)-(n+1)x}, \quad n \geq 0.$$



Repeated composition shifts the graph up less and less and brings the curves closer together.